Spectral Moments of Correlated Wishart Matrices

Zdzisław Burda,* Jerzy Jurkiewicz,† and Bartłomiej Wacław‡
Mark Kac Center for Complex Systems Research and Marian Smoluchowski Institute of Physics, Jagellonian University, ul. Reymonta 4, 30-059 Krakow, Poland

We present an analytic method to determine spectral properties of the covariance matrices constructed of correlated Wishart random matrices. The method gives, in the limit of large matrices, exact analytic relations between the spectral moments and the eigenvalue densities of the covariance matrices and their estimators. The results can be used in practice to extract information about the genuine correlations from the given experimental realization of random matrices.

Wishart random matrices play an important role in the multivariate statistical analysis [1, 2]. They are useful in some problems of fundamental physics [3], communication and information theory [4, 5, 6], internet trading [7] and quantitative finance [8, 9, 10].

A Wishart ensemble of correlated random matrices is defined by a Gaussian probability measure:

$$P(\mathbf{X})D\mathbf{X} = \mathcal{N}^{-1} \exp\left[-\frac{1}{2}\mathrm{Tr}\,\mathbf{X}^\tau \mathbf{C}^{-1}\mathbf{X}\mathbf{A}^{-1}\right] \prod_{i,\alpha=1}^{N,T} dX_{i\alpha}\ , \tag{1}$$

where $\mathbf{X} = (X_{i\alpha})$ is a real rectangular matrix of dimension $N \times T$. It has two types of indices: an N-type index runnig over the set $i = 1, \ldots, N$ and a T-type index over $\alpha = 1, \ldots, T$. Throughout the paper the N-type indices will be denoted by Latin letters and the T-type by Greek ones. $\mathbf{X}^\tau$ denotes the transpose of $\mathbf{X}$. The matrices $\mathbf{C} = (C_{ij})$ and $\mathbf{A} = (A_{\alpha\beta})$ are symmetric square matrices of dimensions $N \times N$ and $T \times T$, respectively. They are positive definite. $\mathcal{N}$ is a normalization constant:

$$\mathcal{N} = (2\pi)^{\frac{NT}{2}} (\det \mathbf{C})^{\frac{T}{2}} (\det \mathbf{A})^{\frac{N}{2}}, \tag{2}$$

chosen to have $\int P(\mathbf{X})D\mathbf{X} = 1$.

Let $Q(\mathbf{X})$ be a quantity depending on $\mathbf{X}$. The average of Q over the random matrix ensemble (1) is defined as:

$$\langle Q \rangle = \int Q(\mathbf{X}) P(\mathbf{X}) D\mathbf{X}. \tag{3}$$

In particular, the two-point correlation function is:

$$\langle X_{i\alpha} X_{j\beta} \rangle = C_{ij} A_{\alpha\beta}, \tag{4}$$

as directly follows from the Gaussian integration. In this paper we are interested in the spectral behaviour, the eigenvalue distribution and the spectral moments of the following random matrices:

$$\mathbf{c} = \frac{1}{T}\mathbf{X}\mathbf{X}^\tau \quad , \quad \mathbf{a} = \frac{1}{N}\mathbf{X}^\tau \mathbf{X}\ . \tag{5}$$

These matrices can be used as estimators of the correlation matrices $\mathbf{C}$ and $\mathbf{A}$ if some realizations of random matrices $\mathbf{X}$ are given. We will refer to $\mathbf{c}$ and $\mathbf{a}$ as to covariance matrices or statistically dressed correlation matrices. We will present an analytic method to determine the eigenvalue distribution and the spectral moments of $\mathbf{c}$ and $\mathbf{a}$ in the limit of large matrix size. Another method of calculating the eigenvalue density of correlated Wishart matrices has been recently discussed in [11].

In parallel to (1) one can define a Wishart ensemble of correlated complex matrices:

$$\mathcal{P}(\mathbf{X})D\mathbf{X} = \mathcal{N}^{-1} \exp\left[-\mathrm{Tr}\,\mathbf{X}^\dagger \mathbf{C}^{-1}\mathbf{X}\mathbf{A}^{-1}\right] \prod_{i,\alpha} dX^{re}_{i\alpha} dX^{im}_{i\alpha}. \tag{6}$$

*burda@th.if.uj.edu.pl
†jjurkiew@th.if.uj.edu.pl
‡bwaclaw@th.if.uj.edu.pl

The matrices $\mathbf{C}$ and $\mathbf{A}$ are now Hermitean and positive definite. $\mathbf{X}^\dagger$ denotes the Hermitean conjugate of $\mathbf{X}$. The normalization constant is now $\mathcal{N} = (\pi)^{NT}(\det \mathbf{C})^T(\det \mathbf{A})^N$. In the analysis of the complex ensemble the estimators (5) of the correlation matrices are replaced correspondingly by

$$\mathbf{c} = \frac{1}{T}\mathbf{X}\mathbf{X}^\dagger \quad , \quad \mathbf{a} = \frac{1}{N}\mathbf{X}^\dagger\mathbf{X} \; . \tag{7}$$

Notice that the factor one half in front of the trace in the measure for real matrices (1) is dropped in (6). With this choice of the measure the two-point correlations take a similar form as for real matrices (4):

$$\langle X_{i\alpha} X^*_{j\beta}\rangle = C_{ij}A_{\alpha\beta} \; . \tag{8}$$

The star stands for the complex conjugation. Additionally we also have:

$$\langle X_{i\alpha} X_{j\beta}\rangle = \langle X^*_{i\alpha} X^*_{j\beta}\rangle = 0 \; . \tag{9}$$

As a consequence, as we shall discuss towards the end of the paper, the matrices $\mathbf{c}$ and $\mathbf{a}$ (5) in the real ensemble have an identical large N behaviour as the corresponding matrices (7) in the complex ensemble. Since we are interested here only in the large N behaviour it is sufficient to consider one of the two ensembles and draw conclusions for the other. We will focus the presentation on the ensemble of real matrices.

An example of a problem which can be formulated in terms of Wishart random matrices (1) is the following. Imagine that we probe a statistical system of N correlated degrees of freedom by doing T measurements. We store the measured values of the i-th degree of freedom in the α-th measurement in a rectangular matrix $\mathbf{X} = (X_{i\alpha})$. The degrees of freedom as well as the measurements may be correlated. This is expressed by the equation (4), which tells us that the covariance matrix for the correlations between degrees of freedom in the system is $\mathbf{C} = (C_{ij})$ and for the (auto)correlation between measurements is $\mathbf{A} = (A_{\alpha\beta})$. Note that in general the correlations between $X_{i\alpha}$ and $X_{j\beta}$ may have a more complicated form: $\langle X_{i\alpha} X_{j\beta}\rangle = \mathcal{C}_{i\alpha,j\beta}$, where the matrix $\mathcal{C}$ has double indices. Such a situation takes place if the autocorrelations are different for various degrees of freedom. We shall not discuss this case here. Moreover, we shall assume that only Gaussian effects are important for the studied system.

A perfect example of the situation described above is the problem of optimal portfolio assessment - one of the fundamental problems of quantitative finance. The portfolio assessment is based on the knowledge of the covariance matrix $\mathbf{C}$ for stocks returns [12]. In practice, the covariance matrix is estimated from the historical data which are stored in a rectangular matrix representing T historical values of N stocks. Fluctuations of returns are well described by the Gaussian ensemble (1). The estimator of the covariance matrix is given by (5). Another problem of modern financial analysis which can be directly cast into the form (1) is the problem of taste matching [7]. This problem is encountered for instance in the large-scale internet trading.

It is worth mentioning that the random matrix framework may also be used in a statistical description of data generated in Monte Carlo simulations for a system with many degrees of freedom, in particular of data concerning the correlation functions. One frequently encounters such a problem in Monte-Carlo simulations of lattice field theory, where the field is represented by correlated numbers distributed on a lattice. Usually, one is forced to use a dynamical Monte Carlo algorithm to sample such a system. The basic idea standing behind a dynamical algorithm is to generate a Markov chain – a sort of a random walk – in the space of configurations. The degrees of freedom on the lattice as well as the successive configurations are usually correlated. Outside a critical region no long range correlations are observed and the fluctuations can be treated as Gaussian.

Complex random matrices are useful for instance in telecommunication or information theory [4, 5, 6].

Let us come back to the ensemble of real matrices (1). As we mentioned the matrices (5) can be treated as estimators of the correlation matrices $\mathbf{C}$ and $\mathbf{A}$. Indeed, from equation (4) we see that:

$$\langle c_{ij}\rangle = \left\langle \frac{1}{T}\sum_\alpha X_{i\alpha}X_{j\alpha}\right\rangle = M_{\mathbf{A}1}C_{ij}, \tag{10}$$

$$\langle a_{\alpha\beta}\rangle = \left\langle \frac{1}{N}\sum_i X_{i\alpha}X_{i\beta}\right\rangle = M_{\mathbf{C}1}A_{\alpha\beta}, \tag{11}$$

where $M_{\mathbf{C}1} = \frac{1}{N}\mathrm{Tr}\ \mathbf{C}$ and $M_{\mathbf{A}1} = \frac{1}{T}\mathrm{Tr}\ \mathbf{A}$. This notation will be explained later. The last equation tells us that measuring the average of $\mathbf{c}$ over the ensemble (1) we obtain the matrix $\mathbf{C}$ up to a constant. In other words having a realization of random matrices $\mathbf{X}$ we can use (5) to estimate $\mathbf{C}$. Similarly, we can use $\mathbf{a}$ to estimate $\mathbf{A}$. Notice that the measure (1) is invariant under the transformation $\mathbf{C} \to b\,\mathbf{C}$ and $\mathbf{A} \to b^{-1}\mathbf{A}$, where b is an arbitrary positive real number. In particular $\langle c_{ij}\rangle$ and $\langle a_{\alpha\beta}\rangle$ are independent of the rescaling factor b. This independence is ensured by

the presence of the factors $M_{\mathbf{A}1}$ and $M_{\mathbf{C}1}$ in equations (10,11). In practical calculations, if $\mathrm{Tr}\,\mathbf{A}$ and $\mathrm{Tr}\,\mathbf{C}$ are not specified, one can remove the redundancy with respect to the rescaling by b, setting $\frac{1}{T}\mathrm{Tr}\,\mathbf{A} = \frac{1}{N}\mathrm{Tr}\,\mathbf{C}$. In this case the constants $M_{\mathbf{A}1} = M_{\mathbf{C}1}$ can be determined from the data by evaluating the traces of $\mathbf{c}$ or $\mathbf{a}$:

$$M_{\mathbf{C}1} = M_{\mathbf{A}1} = \sqrt{\frac{1}{N}\mathrm{Tr}\,\langle \mathbf{c} \rangle} = \sqrt{\frac{1}{T}\mathrm{Tr}\,\langle \mathbf{a} \rangle} \ .$$

While considering the covariance matrices for the Wishart ensemble we can formulate two reciprocal problems, which we shall call **direct** and **inverse problem**. In the direct problem we want to learn as much as possible about the probability distribution of the estimators $\mathbf{c}$ and $\mathbf{a}$ (5) assuming that the matrices $\mathbf{C}$ and $\mathbf{A}$ are given. In particular, we want to calculate the eigenvalue density functions:

$$\rho_{\mathbf{c}}(\lambda) = \left\langle \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle \ ,$$
$$\rho_{\mathbf{a}}(\lambda) = \left\langle \frac{1}{T}\sum_{\alpha=1}^{T} \delta(\lambda - \lambda_\alpha) \right\rangle \ ,$$

where λ_i and λ_α are eigenvalues of $\mathbf{c}$ and $\mathbf{a}$, respectively. The determination of the eigenvalue density functions is equivalent to the determination of all their spectral moments:

$$m_{\mathbf{c}k} = \int d\lambda\, \rho_{\mathbf{c}}(\lambda)\, \lambda^k = \left\langle \frac{1}{N}\mathrm{Tr}\,\mathbf{c}^k \right\rangle \ ,$$
$$m_{\mathbf{a}k} = \int d\lambda\, \rho_{\mathbf{a}}(\lambda)\, \lambda^k = \left\langle \frac{1}{T}\mathrm{Tr}\,\mathbf{a}^k \right\rangle \ .$$

The moments $m_{\mathbf{c}k}, m_{\mathbf{a}k}$ are related to each other:

$$m_{\mathbf{a}k} = r^{1-k} m_{\mathbf{c}k} \ , \tag{12}$$

where $r = N/T$, as follows from the cyclicity of the trace:

$$\mathrm{Tr}\,\mathbf{X}\mathbf{X}^\tau \cdots \mathbf{X}\mathbf{X}^\tau = \mathrm{Tr}\,\mathbf{X}^\tau\mathbf{X} \cdots \mathbf{X}^\tau\mathbf{X} \ .$$

In the inverse problem we want to learn as much as possible about the genuine correlations in the system, which are given by $\mathbf{C}$ and $\mathbf{A}$, using a measured sample of random matrices $\mathbf{X}$. We can do this by computing the estimators $\mathbf{c}$ and $\mathbf{a}$ (5) and relating them to matrices $\mathbf{C}, \mathbf{A}$. In particular we would like to estimate the eigenvalue distributions and the moments of $\mathbf{C}, \mathbf{A}$:

$$M_{\mathbf{C}k} = \frac{1}{N}\mathrm{Tr}\ \mathbf{C}^k = \frac{1}{N}\sum_{i=1}^{N} \Lambda_i^k \ ,$$
$$M_{\mathbf{A}k} = \frac{1}{T}\mathrm{Tr}\ \mathbf{A}^k = \frac{1}{T}\sum_{\alpha=1}^{T} \Lambda_\alpha^k \ ,$$

where Λ_i and Λ_α are eigenvalues of $\mathbf{C}$ and $\mathbf{A}$, respectively. The inverse problem is very important for practical applications, since in practice it is very common to reconstruct the properties of the underlying system from the experimental data.

In the analysis of the spectral properties of the matrices $\mathbf{a}$ and $\mathbf{c}$ it is convenient to apply the Green's function technique. One can define Green's functions for the correlation matrix $\mathbf{C}$ and its statistically fluctuating counterpart $\mathbf{c}$:

$$\mathbf{G}_{\mathbf{C}}(z) = \frac{1}{z\mathbf{1}_N - \mathbf{C}} \ , \tag{13}$$
$$\mathbf{g}_{\mathbf{c}}(z) = \left\langle \frac{1}{z\mathbf{1}_N - \mathbf{c}} \right\rangle , \tag{14}$$

and correspondingly $\mathbf{G}_{\mathbf{A}}(z)$ and $\mathbf{g}_{\mathbf{a}}(z)$ for $\mathbf{A}$ and $\mathbf{a}$. The symbol $\mathbf{1}_N$ stands for the $N \times N$ identity matrix. A corresponding symbol $\mathbf{1}_T$ appears in the definition of $\mathbf{G}_{\mathbf{A}}(z)$ and $\mathbf{g}_{\mathbf{a}}(z)$. The Green's functions are related to the

generating functions for the moments:

$$M_{\mathbf{C}}(z) = \sum_{k=1}^{\infty} \frac{M_{\mathbf{C}k}}{z^k} = \frac{1}{N}\mathrm{Tr}\left(z\mathbf{G}_{\mathbf{C}}(z)\right) - 1 \ , \tag{15}$$

$$m_{\mathbf{c}}(z) = \sum_{k=1}^{\infty} \frac{m_{\mathbf{c}k}}{z^k} = \frac{1}{N}\mathrm{Tr}\left(z\mathbf{g}_{\mathbf{c}}(z)\right) - 1 \ , \tag{16}$$

or inversely:

$$\begin{aligned} \frac{1}{N}\mathrm{Tr}\,\mathbf{G}_{\mathbf{C}}(z) &= \frac{1+M_{\mathbf{C}}(z)}{z} \ , \\ \frac{1}{N}\mathrm{Tr}\,\mathbf{g}_{\mathbf{c}}(z) &= \frac{1+m_{\mathbf{c}}(z)}{z} \ . \end{aligned} \tag{17}$$

The analogous relations exist for $\mathbf{G}_{\mathbf{A}}(z)$ and $\mathbf{g}_{\mathbf{a}}(z)$. The Green's functions can be used for finding the densities of eigenvalues:

$$\rho_{\mathbf{c}}(\lambda) = -\frac{1}{\pi}\mathrm{Im}\,\frac{1}{N}\mathrm{Tr}\,\mathbf{g}_{\mathbf{c}}(\lambda + i0^+) = -\frac{1}{\pi}\mathrm{Im}\,\frac{1+m_{\mathbf{c}}(\lambda+i0^+)}{\lambda+i0^+} \ , \tag{18}$$

and similarly for $\rho_{\mathbf{a}}(\lambda)$. The eigenvalue densities $\rho_{\mathbf{a}}(\lambda)$ and $\rho_{\mathbf{c}}(\lambda)$ are not independent. As follows from (12) the corresponding generating functions (16) fulfill the equation:

$$m_{\mathbf{a}}(z) = r m_{\mathbf{c}}(rz) \ . \tag{19}$$

Combining the last equation with (17) we obtain:

$$\frac{1}{T}\mathrm{Tr}\ \mathbf{g}_{\mathbf{a}}(z) = r^2\frac{1}{N}\mathrm{Tr}\ \mathbf{g}_{\mathbf{c}}(rz) + \frac{1-r}{z} \ . \tag{20}$$

Applying now (18) we have:

$$\rho_{\mathbf{a}}(\lambda) = r^2\rho_{\mathbf{c}}(r\lambda) + (1-r)\delta(\lambda) \ . \tag{21}$$

The meaning of the last term on the right hand side of this equation is that there are $T-N$ zero modes in the matrix $\mathbf{a}$ if $T > N$. The zero modes disappear when $N = T$. Moving the term containing the delta function to the other side of equation, dividing both sides of the equation by r^2 and substituting the parameter r by $s = T/N = 1/r$ we obtain:

$$\rho_{\mathbf{c}}(\lambda) = s^2\rho_{\mathbf{a}}(s\lambda) + (1-s)\delta(\lambda) \ . \tag{22}$$

Therefore, for $T < N$ the zero modes appear in the spectrum $\rho_{\mathbf{c}}(\lambda)$. In this case it is more convenient to use the parameter $s = 1/r$ instead of r. The zero modes appear in the eigenvalue distribution of either $\mathbf{a}$ or $\mathbf{c}$. The two equations (21) and (22) are dual to each other. For $r = s = 1$ they are identical. Because of the duality it is sufficient to solve the problem for $r \leq 1$. We will present a solution for the limit $r = N/T = \mathrm{const} \leq 1$ and $N \to \infty$ neglecting effects of the order $1/N$.

Using a diagrammatic method [13, 14, 15] one can write down a closed set of equations for the Green's function $\mathbf{g}_{\mathbf{c}}(z)$ (14):

$$\begin{aligned} \mathbf{g}_{\mathbf{c}}(z) &= \frac{1}{z\mathbf{1}_N - \mathbf{\Sigma}_{\mathbf{c}}(z)} \quad , \quad \mathbf{g}_{*\mathbf{c}}(z) = \frac{1}{T\mathbf{1}_T - \mathbf{\Sigma}_{*\mathbf{c}}(z)} \ , \\ \mathbf{\Sigma}_{\mathbf{c}}(z) &= \mathbf{C}\mathrm{Tr}\left(\mathbf{A}\,\mathbf{g}_{*\mathbf{c}}(z)\right) \quad , \quad \mathbf{\Sigma}_{*\mathbf{c}}(z) = \mathbf{A}\mathrm{Tr}\left(\mathbf{C}\,\mathbf{g}_{\mathbf{c}}(z)\right) \ . \end{aligned} \tag{23}$$

The set contains four equation for four unknown matrices including $\mathbf{g}_{\mathbf{c}}(z)$ which we want to calculate, and three auxiliary ones: $\mathbf{g}_{*\mathbf{c}}(z)$, $\mathbf{\Sigma}_{\mathbf{c}}(z)$, $\mathbf{\Sigma}_{*\mathbf{c}}(z)$ (see Appendix 1). Each of them can be interpreted in terms of a generating function for appropriately weighted diagrams with two external lines: $\mathbf{g}_{\mathbf{c}}(z), \mathbf{g}_{*\mathbf{c}}(z)$ for all diagrams and $\mathbf{\Sigma}_{\mathbf{c}}(z), \mathbf{\Sigma}_{*\mathbf{c}}(z)$ for one-line-irreducible diagrams [13, 14, 15] (see Appendix 1). In the limit $N \to \infty$ the weights of non-planar diagrams vanish at least as $1/N$. Thus in this limit only planar diagrams give a contribution to the Green's function. Therefore the large N limit is alternatively called the planar limit. The diagrammatic equations (23) hold only in this limit. An

analogous set of equations can be written for the Green's function $\mathbf{g_a}(z)$. The equations are identical to those of (23) if one exchanges $\mathbf{a} \longleftrightarrow \mathbf{c}$, $\mathbf{A} \longleftrightarrow \mathbf{C}$ and $T \longleftrightarrow N$. The two sets can be solved independently of each other. However, as follows from the duality (20) it is sufficient to solve only one of them and deduce the solution of the other.

The equations (23) can be solved for $\mathbf{g_c}(z)$ by a successive elimination of $\mathbf{g}_{*\mathbf{c}}(z)$, $\mathbf{\Sigma_c}(z)$ and $\mathbf{\Sigma}_{*\mathbf{c}}(z)$. However, the resulting equation is very entangled [14]:

$$\mathbf{g_c}(z) = \left(z\mathbf{1}_N - \mathbf{C}\mathrm{Tr}\,\frac{\mathbf{A}}{\mathbf{1}_T T - \mathbf{A}\mathrm{Tr}\,(\mathbf{C}\mathbf{g_c}(z))} \right)^{-1}, \tag{24}$$

and cannot be easily used in practical calculations of the moments $m_{\mathbf{c}k}$ or spectral density $\rho_{\mathbf{c}}(\lambda)$.

Another way of solving the equations (23) was proposed in [15]. It relies on introducing a new complex variable Z conjugate to z which is defined by the equation:

$$m_{\mathbf{c}}(z) = M_{\mathbf{C}}(Z)\ . \tag{25}$$

At the first glance this equation looks useless because it refers to an unknown function $m_{\mathbf{c}}(z)$ which we actually want to determine. Quite contrary to this, as we shall see, the introduction of the conjugate variable Z allows us to write down a closed functional equation for $m_{\mathbf{c}}(z)$. First, let us illustrate how the method works for $\mathbf{A} = \mathbf{1}_T$ [15]. In this case, the elimination of the auxiliary functions (23) leads to

$$Z = \frac{z}{1 + r m_{\mathbf{c}}(z)}\ , \tag{26}$$

or equivalently to

$$z = Z \cdot (1 + r M_{\mathbf{C}}(Z))\ . \tag{27}$$

Suppose we solve the direct problem. In this case we know the matrix $\mathbf{C}$ and hence also the generating function $M_{\mathbf{C}}(Z)$. Inserting (26) to (25) we obtain a closed compact functional relation for $m_{\mathbf{c}}(z)$:

$$m_{\mathbf{c}}(z) = M_{\mathbf{C}}\left(\frac{z}{1 + r m_{\mathbf{c}}(z)} \right). \tag{28}$$

If $M_{\mathbf{C}}(z)$ has a simple form, one can solve the equation for $m_{\mathbf{c}}(z)$ analytically [15]. In general, one can write a numerical program to calculate the eigenvalue density $\rho_{\mathbf{c}}(\lambda)$ from the last equation. In case of solving the inverse problem, we assume that we can determine moments $m_{\mathbf{c}k}$ from the data and hence that we can approximate the generating function $m_{\mathbf{c}}(z)$. Then we can insert (27) to (25) and obtain a functional equation for $M_{\mathbf{C}}(Z)$:

$$M_{\mathbf{C}}(Z) = m_{\mathbf{c}}\left(Z \cdot (1 + r M_{\mathbf{C}}(Z))\right). \tag{29}$$

The problem is solved in principle. However, in practical terms the inverse problem is much more difficult, because one cannot compute all experimental moments $m_{\mathbf{c}k}$ with an arbitrary accuracy, unless one has an infinitely long series of measurements. But one never has. In practice one can estimate only a few lower moments $m_{\mathbf{c}k}$ with a good accuracy. Because of this practical limitation one cannot entirely solve the inverse problem. However, as we discussed in [10] the inverse problem can be partially solved even in specific practical applications using a moments method. Let us sketch this method below.

We can gain some insight into the spectral properties of the correlation matrix $\mathbf{C}$ by determining the relation between the moments $M_{\mathbf{C}k}$ and $m_{\mathbf{c}k}$. Expanding the functions $M_{\mathbf{C}}(z)$ and $m_{\mathbf{c}}(z)$ in (28) in $1/z$ using (15,16) and comparing the coefficients at $1/z^k$ we obtain:

$$\begin{aligned}
m_{\mathbf{c}1} &= M_{\mathbf{C}1} \\
m_{\mathbf{c}2} &= M_{\mathbf{C}2} + r M_{\mathbf{C}1}^2 \\
m_{\mathbf{c}3} &= M_{\mathbf{C}3} + 3r M_{\mathbf{C}1} M_{\mathbf{C}2} + r^2 M_{\mathbf{C}1}^3 \\
m_{\mathbf{c}4} &= M_{\mathbf{C}4} + 2r\left(M_{\mathbf{C}2}^2 + 2 M_{\mathbf{C}1} M_{\mathbf{C}3}\right) + 6r^2 M_{\mathbf{C}1}^2 M_{\mathbf{C}2} + r^3 M_{\mathbf{C}1}^4 \\
&\ldots
\end{aligned} \tag{30}$$

We can also invert the equations for $M_{\mathbf{C}k}$. The result of inversion gives a set of equations which can be directly obtained from the $1/Z$ expansion of the functions in the equation (29) which is the inverse transform of (28). We can also determine the corresponding relations for 'negative' moments $m_{\mathbf{c}k}$ and $M_{\mathbf{C}k}$, that is for $k < 0$, or determine the spectral density $\rho_{\mathbf{c}}(\lambda)$ [15]. Using a computer tool for symbolic calculations one can easily write a program which successively generates the relations between spectral moments (30) from the equation (28).

The calculations get more complicated in the general case when both $\mathbf{C}$ and $\mathbf{A}$ are arbitrary. The guiding principle is the same, though. We introduce the conjugate variable Z (25) and, using it, write down the solution of the equations (23). In the direct problem we assume that the generating functions $M_{\mathbf{C}}(Z)$ and $M_{\mathbf{A}}(Z)$ are known. We will show that in this case the solution of (23) takes a form of an explicit equation for $z = z(Z)$, where the function $z(Z)$ depends on the functions $M_{\mathbf{C}}(Z)$ and $M_{\mathbf{A}}(Z)$. Inserting this solution back to (23) we eventually obtain a functional equation $m_{\mathbf{c}}(z(Z)) = M_{\mathbf{C}}(Z)$ from which we can extract the function $m_{\mathbf{c}}(z)$.

The solution of (23) takes the form:

$$\frac{z}{Z} = \frac{1}{T}\sum_{\alpha=1}^{T}\frac{\Lambda_\alpha}{1-\Lambda_\alpha r\frac{Z}{z}M_{\mathbf{C}}(Z)}, \tag{31}$$

where Λ_α are eigenvalues of $\mathbf{A}$. It can be rewritten as:

$$rM_{\mathbf{C}}(Z) = M_{\mathbf{A}}\left(\frac{z}{rZM_{\mathbf{C}}(Z)}\right) . \tag{32}$$

and can be formally solved for z:

$$z = Z\, rM_{\mathbf{C}}(Z)\, M_{\mathbf{A}}^{-1}\left(rM_{\mathbf{C}}(Z)\right), \tag{33}$$

where $M_{\mathbf{A}}^{-1}$ is the inverse function of $M_{\mathbf{A}}$. Thus we have obtained an explicit equation for $z = z(Z)$ in terms of the known functions $M_{\mathbf{A}}$ and $M_{\mathbf{C}}$. One can easily check that for $\mathbf{A} = \mathbf{1}_T$ the last equation reduces to (27). In this case $M_{\mathbf{A}}(z) = 1/(z-1)$, $M_{\mathbf{A}}^{-1}(z) = 1 + 1/z$.

Combining the equation for $z = z(Z)$ given by (33) with (25) we arrive at a closed equation for the generating function $m_{\mathbf{c}}(Z)$. It can be used for example to calculate the moments $m_{\mathbf{c}k}$'s (see Appendix 2). The calculations yield a set of equations expressing $m_{\mathbf{c}k}$'s in terms of the bare moments $M_{\mathbf{A}k}$ and $M_{\mathbf{C}k}$:

$$\begin{aligned}
m_{\mathbf{c}1} &= M_{\mathbf{C}1}M_{\mathbf{A}1}\\
m_{\mathbf{c}2} &= M_{\mathbf{C}2}M_{\mathbf{A}1}^2 + rM_{\mathbf{C}1}^2M_{\mathbf{A}2}\\
m_{\mathbf{c}3} &= M_{\mathbf{C}3}M_{\mathbf{A}1}^3 + 3rM_{\mathbf{C}1}M_{\mathbf{C}2}M_{\mathbf{A}1}M_{\mathbf{A}2} + r^2M_{\mathbf{C}1}^3M_{\mathbf{A}3}\\
m_{\mathbf{c}4} &= M_{\mathbf{C}4}M_{\mathbf{A}1}^4 + 2r\left(M_{\mathbf{C}2}^2 + 2M_{\mathbf{C}1}M_{\mathbf{C}3}\right)M_{\mathbf{A}1}^2M_{\mathbf{A}2} + 2r^2M_{\mathbf{C}1}^2M_{\mathbf{C}2}\left(M_{\mathbf{A}2}^2 + 2M_{\mathbf{A}1}M_{\mathbf{A}3}\right) + r^3M_{\mathbf{C}1}^4M_{\mathbf{A}4}\\
&\dots
\end{aligned} \tag{34}$$

The equations reduce to the form (30) for $\mathbf{A} = \mathbf{1}_T$.

Using the relations (12) we can also determine the moments of the matrix $\mathbf{a}$. It is more convenient to write them using the variable $s = r^{-1}$ – the dual counterpart of r – instead of r itself:

$$\begin{aligned}
m_{\mathbf{a}1} &= M_{\mathbf{A}1}M_{\mathbf{C}1}\\
m_{\mathbf{a}2} &= M_{\mathbf{A}2}M_{\mathbf{C}1}^2 + sM_{\mathbf{A}1}^2M_{\mathbf{C}2}\\
m_{\mathbf{a}3} &= M_{\mathbf{A}3}M_{\mathbf{C}1}^3 + 3sM_{\mathbf{A}1}M_{\mathbf{A}2}M_{\mathbf{C}1}M_{\mathbf{C}2} + s^2M_{\mathbf{A}1}^3M_{\mathbf{C}3}\\
m_{\mathbf{a}4} &= M_{\mathbf{A}4}M_{\mathbf{C}1}^4 + 2s\left(M_{\mathbf{A}2}^2 + 2M_{\mathbf{A}1}M_{\mathbf{A}3}\right)M_{\mathbf{C}1}^2M_{\mathbf{C}2} + 2s^2M_{\mathbf{A}1}^2M_{\mathbf{A}2}\left(M_{\mathbf{C}2}^2 + 2M_{\mathbf{C}1}M_{\mathbf{C}3}\right) + s^3M_{\mathbf{A}1}^4M_{\mathbf{C}4}\\
&\dots
\end{aligned} \tag{35}$$

The equations are completely symmetric to (34) with respect to the change $r \longleftrightarrow s$ (which amounts to $N \longleftrightarrow T$), and $\mathbf{c} \longleftrightarrow \mathbf{a}$, $\mathbf{C} \longleftrightarrow \mathbf{A}$. Using this method one can obtain equations (34) and (35) to an arbitrary order.

The above relations are useful for computing the dressed moments $m_{\mathbf{a}k}, m_{\mathbf{c}k}$ for given matrices $\mathbf{A}, \mathbf{C}$ or inversely, the genuine moments $M_{\mathbf{A}k}, M_{\mathbf{C}k}$ from the experimental data. As mentioned, the spectral moments give us in principle full information about the eigenvalue distribution. In practice the reconstruction of the eigenvalue density may be difficult, because to do it we would need to know all moments with a very good precision. Usually, in practical applications one can accurately evaluate only a few lower moments.

In some special cases if we can make some extra assumptions about the form of the matrices $\mathbf{C}$ or $\mathbf{A}$ we can improve significantly the reconstruction of the eigenvalue density. In the previous work [15] we have analysed the case of $\mathbf{A} = \mathbf{1}_T$ and of the matrix $\mathbf{C}$ which had only a few distinct eigenvalues. In this case the Green's function $\mathbf{g}_{\mathbf{c}}(z)$ is given by an algebraic equation of the order which is equal to the number of distinct eigenvalues. It can be analytically solved when this number is less or equal four. If it is larger the problem can be handled numerically. The duality tells us that the solution also holds when we change the roles of $\mathbf{A}$ and $\mathbf{C}$.

Below we will discuss the case of exponential autocorrelations. Exponential correlations are encountered in many situations. The general solution, which we have discussed so far, simplifies in this case to a more compact relation

for the Green's function, which allows us to find analytically an approximate form of the eigenvalue density of the random matrices $\mathbf{c}$ and $\mathbf{a}$. The approximation becomes exact in the large N limit. We consider purely exponential autocorrelations given by the autocorrelation matrix:

$$A_{\alpha\beta} = \exp\left[-|\alpha-\beta|/t\right] \ , \tag{36}$$

where t controls the range of autocorrelations. The inverse of the matrix $\mathbf{A}$ reads:

$$\mathbf{A}^{-1} = \frac{1}{2sh}\begin{bmatrix} ex, & -1, & & & & \\ -1, & 2ch, & -1, & & & \\ & \cdots & \cdots & \cdots & & \\ & & -1, & 2ch, & -1 & \\ & & & -1, & ex \end{bmatrix} . \tag{37}$$

We have introduced here a shorthand notation $ex = \exp(1/t)$, $ch = \cosh(1/t)$ and $sh = \sinh(1/t)$. The spectrum of this matrix can be approximated by the spectrum of a matrix $\mathbf{M}$:

$$\mathbf{M} = \frac{1}{2sh}\begin{bmatrix} 2ch, & -1, & & & & -1 \\ -1, & 2ch, & -1, & & & \\ & \cdots & \cdots & \cdots & & \\ & & -1, & 2ch, & -1 & \\ -1, & & & -1, & 2ch & \end{bmatrix} , \tag{38}$$

whose eigenvalues can be found analytically:

$$\mu_\alpha = (ch + \cos(\pi\alpha/T))/sh.$$

The corresponding eigenvectors are given by the Fourier modes. The matrix $2sh \cdot \mathbf{M}$ can be viewed as of a sum: $(2ch-2)\mathbf{1}_T + \Delta_T$, of a unity matrix multiplied by a constant and a discretized one-dimensional Laplacian Δ_T for a cyclic chain of length T. The matrix $\mathbf{A}^{-1}$ can be obtained from $\mathbf{M}$ by adding to it a perturbation $\mathbf{P}$: $\mathbf{A}^{-1} = \mathbf{M} + \mathbf{P}$, where $\mathbf{P}$ has only four non-vanishing elements: $P_{11} = P_{TT} = ex^{-1}$ and $P_{1T} = P_{T1} = 1$. The first order corrections to the eigenvalues of $\mathbf{A}^{-1}$, which stem from the perturbation $\mathbf{P}$, behave as $1/T$. The perturbation $\mathbf{P}$ can be viewed as a change of a boundary condition of the Laplacian. As usual, boundary conditions affect mostly the longest (small momentum) modes. Indeed, a careful analysis shows that the two diagonal terms of the perturbation matrix, $P_{11} = P_{TT}$, introduce a constant correction independent of T of the lowest eigenvalues which does not vanish when T goes to infinity. However, since the differences between unperturbed eigenvalues of $\mathbf{M}$ and the corresponding perturbed eigenvalues of $\mathbf{A}^{-1}$ disappear for all other eigenvalues, we expect that for $t \ll T$ the spectral properties of $\mathbf{A}$ can be well approximated by the eigenvalues of $\mathbf{M}^{-1}$:

$$\Lambda_\alpha \approx \frac{1}{\mu_\alpha} = \frac{sh}{ch + \cos(\pi\alpha/T)} \ . \tag{39}$$

In this limit we can also approximate the sum (31) by an integral:

$$\frac{z}{Z} \approx \frac{sh}{\pi}\int_0^\pi \frac{d\tau}{(ch - sh\cdot F) + \cos\tau} \ , \tag{40}$$

Where $\tau = \pi\alpha/T$ and the symbol $F \equiv \frac{Z}{z} r m_{\mathbf{c}}(z)$ is introduced for brevity. Note that in the definition of F we have replaced $M_{\mathbf{C}}(Z)$, which would rather be dictated by (31), by $m_{\mathbf{c}}(z)$. This change is legitimate due to (25). The integral (40) can be done:

$$\frac{z}{Z} = \frac{sh}{\sqrt{(ch - sh\,F)^2 - 1}} \ . \tag{41}$$

Setting back $F = \frac{Z}{z} r m_{\mathbf{c}}(z)$ we eventually obtain:

$$Z = z\frac{ch\cdot r m_{\mathbf{c}}(z) - \sqrt{sh^2 + r^2 m_{\mathbf{c}}^2(z)}}{sh\cdot(r^2 m_{\mathbf{c}}^2(z) - 1)} \ . \tag{42}$$

$t=1$			
T	$M_{\mathbf{A}2}$	$M_{\mathbf{A}3}$	$M_{\mathbf{A}4}$
20	1.29493	2.01479	3.48620
50	1.30579	2.05757	3.60838
100	1.30941	2.07183	3.64911
200	1.31123	2.07896	3.66947
500	1.31231	2.08324	3.68169
∞	1.31304	2.08609	3.68983

$t=5$			
T	$M_{\mathbf{A}2}$	$M_{\mathbf{A}3}$	$M_{\mathbf{A}4}$
20	4.44996	28.6455	204.107
50	4.81980	34.2544	271.932
100	4.94314	36.1292	294.733
200	5.00482	37.0666	306.133
500	5.04182	37.6290	312.973
∞	5.06649	38.0040	317.534

$t=10$			
T	$M_{\mathbf{A}2}$	$M_{\mathbf{A}3}$	$M_{\mathbf{A}4}$
20	7.58726	79.6134	891.336
50	9.03668	120.509	1784.56
100	9.53497	135.501	2146.93
200	9.78414	143.001	2328.48
500	9.93364	147.501	2437.40
∞	10.0333	150.501	2510.02

TABLE I: The moments $M_{\mathbf{A}2}$, $M_{\mathbf{A}3}$ and $M_{\mathbf{A}4}$ of the matrix $\mathbf{A}$ (36) for three values of the autocorrelation length $t = 1, 5, 10$ calculated numerically for finite size $T = 20, \ldots, 500$ and by the analytic formula (45) which corresponds to $T = \infty$. The finite size values approach the values given by (45) as T/t tends to infinity.

This is an explicit equation for $Z = Z(z)$ which can be now inserted into $m_{\mathbf{c}}(z) = M_{\mathbf{C}}(Z)$ giving us a compact relation for $m_{\mathbf{c}}(z)$ in the presence of the exponential autocorrelations (36):

$$m_{\mathbf{c}}(z) = M_{\mathbf{C}}\left(z\frac{ch\cdot r m_{\mathbf{c}}(z) - \sqrt{sh^2 + r^2 m_{\mathbf{c}}^2(z)}}{sh\cdot(r^2 m_{\mathbf{c}}^2(z) - 1)}\right) . \tag{43}$$

In the limit $t \to 0$, the parameters $sh = \sinh(1/t)$, $ch = \cosh(1/t)$ and $ex = \exp(1/t)$ increase to infinity and $ch/ex \approx sh/ex \approx 1$. As a consequence, the form of equation (42) simplifies to (26), which corresponds to the case without autocorrelations, as expected. Using the equation (43) we can recursively generate equations for the consecuitive moments:

$$\begin{aligned}
&m_{\mathbf{c}1} = M_{\mathbf{C}1}\\
&m_{\mathbf{c}2} = M_{\mathbf{C}2} + rM_{\mathbf{C}1}^2\cdot cth\\
&m_{\mathbf{c}3} = M_{\mathbf{C}3} + 3rM_{\mathbf{C}1}M_{\mathbf{C}2}\cdot cth + r^2M_{\mathbf{C}1}^3\left(\frac{3}{2}\,cth^2 - \frac{1}{2}\right)\\
&m_{\mathbf{c}4} = M_{\mathbf{C}4} + 2r(M_{\mathbf{C}2}^2 + 2M_{\mathbf{C}1}M_{\mathbf{C}3})\cdot cth + 2r^2M_{\mathbf{C}1}^2M_{\mathbf{C}2}\left(4cth^2 - 1\right) + r^3M_{\mathbf{C}1}^4\left(\frac{5}{2}cth^3 - \frac{3}{2}cth\right)\\
&\cdots ,
\end{aligned} \tag{44}$$

where $cth = ch/sh = \coth(1/t)$. The coefficients on the right hand side, which depend on cth, can be directly expressed in terms of the moments $M_{\mathbf{A}k}$ of the matrix $\mathbf{A}$. Approximating again a sum by an integral in the large T limit we can write:

$$M_{\mathbf{A}k} = \frac{1}{T}\sum_{\alpha=1}^{T}\Lambda_\alpha^k \approx \frac{sh^k}{\pi}\int_0^\pi \frac{d\tau}{(ch+\cos\tau)^k} .$$

The integrals can be calculated yielding:

$$\begin{aligned}
M_{\mathbf{A}1} &= 1\\
M_{\mathbf{A}2} &= cth\\
M_{\mathbf{A}3} &= \tfrac{3}{2}cth^2 - \tfrac{1}{2}\\
M_{\mathbf{A}4} &= \tfrac{5}{2}cth^3 - \tfrac{3}{2}cth\\
&\cdots
\end{aligned} \tag{45}$$

We see that if we insert these coefficients into the equations (34) we obtain (44). This is a consistency check for the approximation which we use here. The quality of this approximation can also be checked by comparing the moments $M_{\mathbf{A}k}$ of the matrix (36) for finite T with the result (45) which corresponds to $T = \infty$. We expect that for $t \ll T$ the numerical values shall approach the result (45). The results of this comparison confirm our expectations (see table 1). Thus we see that the formula (44) for $m_{\mathbf{c}}(z)$ in the presence of the exponential autocorrelations becomes exact in the limit $T \to \infty$. This formula allows us to compute the eigenvalue distribution of the random matrices $\mathbf{c}$ and $\mathbf{a}$ (5). Let

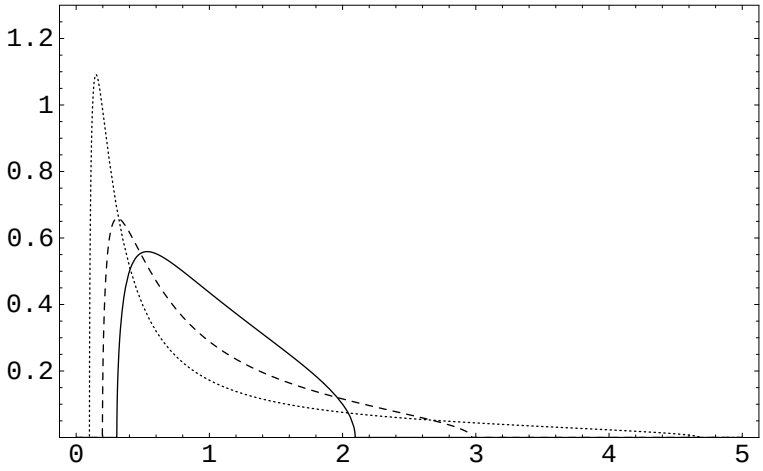

FIG. 1: The density of eigenvalues $\rho_{\mathbf{c}}(\lambda)$ for exponential matrix $\mathbf{A}$, $\mathbf{C} = \mathbf{1}_N, r = 0.2$ and for three different autocorrelation times t: $t = 0$ (solid line), $t = 2$ (dashed line) and $t = 5$ (dotted line).

us illustrate this on the simplest example of the system which has no correlations: $\mathbf{C} = \mathbf{1}_N$ and $M_{\mathbf{C}}(Z) = 1/(Z-1)$. In this case the equation (43) for $m_{\mathbf{c}}(z)$ takes the form:

$$cth \cdot x^2 - \frac{1}{z}(x^2-1)(x+r) - x\sqrt{1+x^2/sh^2} = 0 \ , \tag{46}$$

where we have used the notation $x(z) = rm_{\mathbf{c}}(z)$. For $t \to 0$ ($\mathbf{A} \to \mathbf{1}_T$) this equation reduces to:

$$(x-1)\left[x - \frac{1}{z}(x+1)(x+r)\right] = 0 \ , \tag{47}$$

which has a solution:

$$x(z) = \frac{1}{2}\left(-1-r-i\sqrt{(\mu_+ - z)(z-\mu_-)} + z\right), \tag{48}$$

where $\mu_\pm = (1 \pm \sqrt{r})^2$, which leads to the well-known result for the uncorrelated Wishart ensemble:

$$\rho_{\mathbf{c}}(\lambda) = \frac{1}{2\pi r}\frac{\sqrt{(\mu_+ - \lambda)(\lambda - \mu_-)}}{\lambda} \ . \tag{49}$$

For $t > 0$ it is still possible to find analytically a solution of (46). Let us rewrite (46) as a polynomial equation:

$$-z^2x^2(1+x^2/sh^2) + \left(cth\, zx^2 - (r+x)(x^2-1)\right)^2 = 0 \ .$$

It has two trivial solutions $x = \pm 1$. Dividing out the polynomial $(x-1)(x+1)$ we get:

$$x^4 + 2x^3(r - cth\, z) + x^2(-1 + r^2 - 2cth\, rz + z^2) - 2rx - r^2 = 0 \ . \tag{50}$$

This is a quartic equation which can be solved analytically by the Ferrari method. We will not present the formal solution which is neither transparent nor informative. Instead, we show in Fig. 1 the eigenvalue density functions $\rho_{\mathbf{c}}(\lambda)$, for different t, resulting from this solution. The lower part of the distribution approaches zero when t increases, but zero modes do not appear in the distribution as long as $r < 1$. The formula (43) applies to any correlation matrix $\mathbf{C}$ but in the general case one has to use a numerical procedure to calculate from it the density function.

Let us stop here the presentation of results for the ensemble of real matrices. As we mentioned all results in the large N limit hold also in the ensemble of complex matrices if the covariance matrices (5) are replaced by (7). The reason why it is so is related to the fact that the moments of $\mathbf{c} = \frac{1}{T}\mathbf{X}\mathbf{X}^\dagger$ in the ensemble of complex matrices (6) are equal to the moments of $\mathbf{c} = \frac{1}{T}\mathbf{X}\mathbf{X}^\tau$ in the real ensemble (1) up to a $1/N$ corrections which disappear in the large N limit:

$$\frac{1}{N}\left\langle \left(\frac{1}{T}\mathbf{X}\mathbf{X}^\dagger\right)^k \right\rangle_{\text{complex}} = \frac{1}{N}\left\langle \left(\frac{1}{T}\mathbf{X}\mathbf{X}^\tau\right)^k \right\rangle_{\text{real}} + O(1/N) \tag{51}$$

Let us illustrate this by explicit calculations of the second moment. Using the Wick's theorem for Gaussian integrals and the equation (4) for the two-point correlation function, we have:

$$\begin{aligned}
&\frac{1}{N}\left\langle\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\tau}\right)^{2}\right\rangle = \frac{1}{NT^2}\left\langle X_{i\alpha}X_{j\alpha}X_{j\beta}X_{i\beta}\right\rangle = \\
&= \frac{1}{NT^2}\left\{\left\langle X_{i\alpha}X_{j\alpha}\right\rangle\left\langle X_{j\beta}X_{i\beta}\right\rangle + \left\langle X_{i\alpha}X_{i\beta}\right\rangle\left\langle X_{j\alpha}X_{j\beta}\right\rangle + \left\langle X_{i\alpha}X_{j\beta}\right\rangle\left\langle X_{j\alpha}X_{i\beta}\right\rangle\right\} \\
&= \frac{1}{NT^2}\left\{C_{ij}A_{\alpha\alpha}C_{ji}A_{\beta\beta} + C_{ii}A_{\alpha\beta}C_{jj}A_{\alpha\beta} + C_{ij}A_{\alpha\beta}C_{ji}A_{\alpha\beta}\right\} \\
&= M_{\mathbf{C}2}M_{\mathbf{A}1}^2 + rM_{\mathbf{C}1}^2M_{\mathbf{A}2} + \frac{r}{N}M_{\mathbf{C}2}M_{\mathbf{A}2}\ .
\end{aligned}$$

The corresponding calculations for the complex ensemble read:

$$\begin{aligned}
&\frac{1}{N}\left\langle\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\dagger}\right)^{2}\right\rangle = \frac{1}{NT^2}\left\langle X_{i\alpha}X_{j\alpha}^{*}X_{j\beta}X_{i\beta}^{*}\right\rangle = \\
&= \frac{1}{NT^2}\left\{\left\langle X_{i\alpha}X_{j\alpha}^{*}\right\rangle\left\langle X_{j\beta}X_{i\beta}^{*}\right\rangle + \left\langle X_{i\alpha}X_{i\beta}^{*}\right\rangle\left\langle X_{j\alpha}X_{j\beta}^{*}\right\rangle + \left\langle X_{i\alpha}X_{j\beta}\right\rangle\left\langle X_{j\alpha}^{*}X_{i\beta}^{*}\right\rangle\right\} \\
&= \frac{1}{NT^2}\left\{C_{ij}A_{\alpha\alpha}C_{ji}A_{\beta\beta} + C_{ii}A_{\alpha\beta}C_{jj}A_{\alpha\beta}\right\} \\
&= M_{\mathbf{C}2}M_{\mathbf{A}1}^2 + rM_{\mathbf{C}1}^2M_{\mathbf{A}2}\ .
\end{aligned}$$

The difference between the two calculations appears in the third term which in the real ensemble gives a contribution of the order $1/N$ while in the complex ensemble disappears by virtue of (9). We recognize that $M_{\mathbf{C}2}M_{\mathbf{A}1}^2 + rM_{\mathbf{C}1}^2M_{\mathbf{A}2}$ which are the leading terms in the $1/N$ expansion are identical as in the second equation in the set (34). Generally one can show that the leading contributions which correspond to the planar diagrams in the expansion of $\mathbf{g}_{\mathbf{c}}(z)$ are identical for both ensembles. Non-planar diagrams are different but they contribute in the subleading orders: it turns out that in the diagrammatic expansion of the Green's function $g_c(z)$ for the complex matrix ensemble (6), which would be a counterpart of (52) in the appendix, all diagrams which contain a double arc with dashed and solid line crossed are identically equal zero since such an arc corresponds to the propagator $\langle X_{i\alpha}X_{j\beta}\rangle$ or $\langle X_{i\alpha}^{*}X_{j\beta}^{*}\rangle$. A crossing of two arcs is however allowed and leads to a factor $1/N^2$.

To summarize: in the paper we have considered an Wishart ensemble of correlated random matrices. We have obtained in the limit of large matrices a closed set of equations relating the Green's function or equivalently the moments' generating functions $m_{\mathbf{c}}(z)$ and $m_{\mathbf{a}}(z)$ for statistically dressed correlations to the generating functions for genuine correlation matrices $M_{\mathbf{C}}(z)$ and $M_{\mathbf{A}}(z)$. The equations in the large N limit are the same for the ensemble of real and complex matrices. Using these equations we can write down exact relations between genuine and experimental spectral moments of correlation functions of an arbitrary order. The relations can be used in practical problems to learn about correlations in the studied system from the experimental samples. In the case of exponential correlations we have also found an explicit form the spectral density function of the covariance matrix. A natural generalization of the work presented here is to consider a more general type of time correlations than purely exponential (36). If the correlations are of the form which depends on the time difference $A_{\alpha,\beta} = A(|\alpha - \beta|)$ and if they are short-ranged then one can apply Fourier transform to determine in the large T limit an approximate spectrum of the matrix $\mathbf{A}$ and approximate values of its spectral moments. Another interesting issue which can be addressed in the future is the determination of the probability distribution for individual elements of the covariance matrices $\mathbf{c}$ and $\mathbf{a}$ similarly as it was done for the uncorrelated case [16].

Acknowledgments

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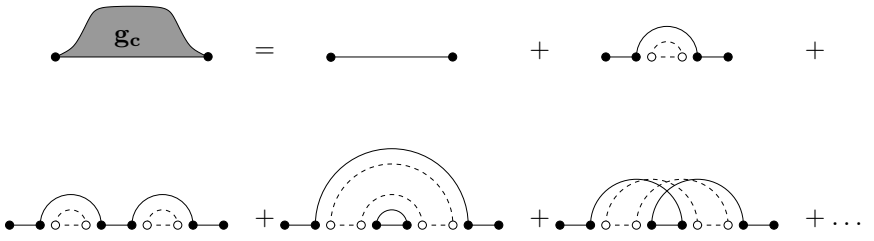
Appendix 1

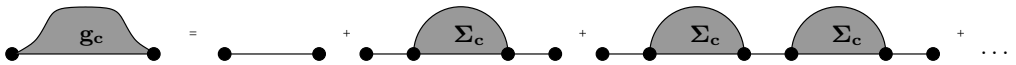
For completeness we recall here the graphical representation of the Green's function. The details of the diagrammatic method can be found in [13, 14, 15]. The Green's function $\mathbf{g_c}(z)$ can be represented as a sum over diagramms:



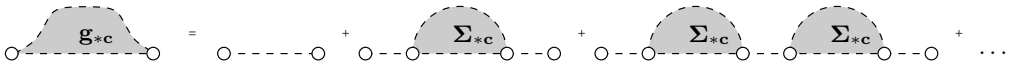
(52)

where the N-type and T-type indices of $\mathbf{X}$ are denoted by filled and empty circles, respectively. The matrix $\mathbf{X}$ is denoted by an ordered pair of neighbouring filled and empty circles, while $\mathbf{X}^\tau$ is drawn as an pair of such circles in the reverse order. A horizontal solid line stands for $\mathbf{1}_N/z$, a dashed line for $\mathbf{1}_T/T$, a solid arc for $\mathbf{C}$ and a dashed arc for $\mathbf{A}$. The two point function (4) is drawn as a double arc. Matrices on a line are multiplied in the order of appearance on this line. If a line is closed, the trace is taken.

In the thermodynamical limit only planar diagrams give contribution to $\mathbf{g_c}$. In particular the last term in (52) vanishes. The Green's function $\mathbf{g_{c*}}(z)$ is represented by an identical set of diagrams with dashed and solid lines exchanged. It is convenient to introduce one-line irreducible diagrams and corresponding generating functions $\mathbf{\Sigma_c}$ and $\mathbf{\Sigma_{*c}}$. The Green's functions can be expressed in terms of $\mathbf{\Sigma_c}$ and $\mathbf{\Sigma_{*c}}$ as follows:



(53)



(54)

In the planar limit there are two additional equations which relate the sums over one-line irreducible diagrams to the Green's functions:

(55)

Analogous diagrammatic equations can be written for $\mathbf{g_a}$ with the only difference that the solid line shall denote the propagator $\mathbf{1}_T/z$ and the dashed line $\mathbf{1}_N/N$.

Appendix 2

We use the equations (33) and (25) to determine the relations (34). As for the case $\mathbf{A} = \mathbf{1}_T$ we shall do this using $1/z$ expansion. The function $M_{\mathbf{A}}(Z)$ is given by the series:

$$M_{\mathbf{A}}(Z) = \frac{M_{\mathbf{A}1}}{Z} + \frac{M_{\mathbf{A}2}}{Z^2} + \frac{M_{\mathbf{A}3}}{Z^3} + \dots \ . \tag{56}$$

Let us determine the expansion for the inverse function $M_{\mathbf{A}}^{-1}$ as a series around zero:

$$M_{\mathbf{A}}^{-1}(y) = M_{\mathbf{A}1} y^{-1} \left(1 + \mu_1 y + \mu_2 y^2 + \dots\right). \tag{57}$$

The coefficients of the series can be directly calculated from the condition:

$$y = M_{\mathbf{A}}(M_{\mathbf{A}}^{-1}(y)) \ , \tag{58}$$

which gives us:

$$\mu_1 = \frac{M_{\mathbf{A}2}}{(M_{\mathbf{A}1})^2} \quad , \quad \mu_2 = \frac{M_{\mathbf{A}3} M_{\mathbf{A}1} - (M_{\mathbf{A}2})^2}{(M_{\mathbf{A}1})^4} \quad , \quad \dots \quad . \tag{59}$$

The equation (33) takes the form:

$$z = M_{\mathbf{A}1} Z \left(1 + \mu_1 r M_{\mathbf{C}}(Z) + \mu_2 r^2 M^2_{\mathbf{C}}(Z) + \dots\right) \ , \tag{60}$$

or if written for $1/z$:

$$\frac{1}{z} = \frac{1}{M_{\mathbf{A}1}} \frac{1}{Z} \left(1 - \mu_1 r M_{\mathbf{C}}(Z) + (\mu_1^2 - \mu_2) r^2 M^2_{\mathbf{C}}(Z) + \dots\right), \tag{61}$$

where:

$$M_{\mathbf{C}}(Z) = \frac{M_{\mathbf{C}1}}{Z} + \frac{M_{\mathbf{C}2}}{Z^2} + \frac{M_{\mathbf{C}3}}{Z^3} + \dots \ . \tag{62}$$

Thus we have expressed $1/z$ as a series of $1/Z$. Inserting this series to the equation (25) and comparing coefficients at $1/Z^k$ we eventually obtain (34).

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